Spin Geometry, by Lawson & Michelsohn

Note by Conan Leung

(I) Clifford alg, Spin(n) & representations

§ $V \cong \mathbb{R}^n$ (or $\mathbb{C}^n$) w/ $q \in \mathrm{Sym}^2 V^*$,

i.e. quadratic form on V.

$$\underset{(\text{or } Cl(V,q))}{Cl_n} := \underbrace{T(V)}_{\sum_k V^{\otimes k}} \Big/ u\otimes v + v \otimes u + 2q(u,v)1$$

- algebra, $\mathbb{Z}_2$-graded ($\because u\otimes v + v\otimes u$ & 1 : even deg.)

- $\underset{(\mathbb{Z}\text{-graded alg.})}{\Lambda^{\bullet} V} \xrightarrow[\text{vector space}]{\cong} Cl(V)$, $q \equiv 0$

  ↳ deform alg. str. on $\Lambda^{\bullet} V$, only $\mathbb{Z}_2$-gr.

§ $O(n) = \mathrm{Aut}\left(\mathbb{R}^n, q = \sum_{i=1}^n x_i^2\right)$

$\left(O(n,\mathbb{C}) = \mathrm{Aut}\left(\mathbb{C}^n, q = \sum_{i=1}^n z_i^2\right)\right.$ & similar for $O(p,q)$)

- $\forall g \in O(n) \Rightarrow g = \left(\begin{array}{cc|c} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ \hline & & \ddots \pm 1 \ddots \end{array}\right)$

rotation = (reflection)^2

$\Longrightarrow g = \prod^{\leq n}(\text{reflection})$ (true $\forall q$ $\forall \mathbb{k}$ w/ char $\neq 2$)

$g \in SO(n) \longleftrightarrow$ # reflections $\in 2\mathbb{Z}$

- reflection w.r.t. v w/ $|v|=1$:

$$u \mapsto u - 2q(u,v)v$$

Recall $u\cdot v + v\cdot u = 2q(u,v)v$

$$\Rightarrow u\cdot \underbrace{v\cdot v}_{-|v|^2} + v\cdot u\cdot v = 2q(u,v)v$$

$$\Rightarrow u - 2q(u,v)v = v\cdot u\cdot v$$

$Spin(n) :=$ Group generated by $v\in V$, $q(v)=1$ inside Cl_n^+ ← even deg.

$$Spin(n) \xrightarrow{\rho} SO(n)$$

$$\varphi \mapsto (u \mapsto \varphi u \varphi^{-1})$$

$v \mapsto$ reflect^n wrt v

i.e. $u \mapsto u - 2\frac{q(v,u)}{q(v,v)}v = v\cdot u\cdot v$

[onto] ✓

[Ker] $\varphi\cdot u\cdot \varphi^{-1} = u \quad \forall u \in V$

i.e. $\varphi u = u\varphi$

Choose o.n. base $v_1,\dots,v_n$ of (V,q)

Write $\varphi = a + v_1 b$ w/ $a, b \sim V_{\geq 2}$

deg: even, odd ($\because$ deg φ : even)

$$\underbrace{a v_1}_{v_1 a} + v_1 \underbrace{b v_1}_{-v_1 b} \overset{\varphi v_1 = v_1 \varphi}{=\!=\!=} v_1 a + v_1\cdot v_1\cdot b$$

($\because$ deg $a \in 2\mathbb{Z}$)

$$\to v_1^2 b = 0 \Rightarrow b = 0$$

i.e. $\varphi \not\sim v_1$. Inductively, $\varphi \not\sim v_2$, ... etc.

i.e. $\varphi \in \mathbb{R}$ (or $\mathbb{C}$) w/ $\varphi^2 = \pm 1$.

i.e. $0 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$

or $0 \to \underbrace{\{\pm 1, \pm i\}}_{\mathbb{Z}_4} \to Spin(n,\mathbb{C}) \to SO(n,\mathbb{C}) \to 1$.

§ $Cl_{r,s} \triangleq Cl(\mathbb{R}^{r,s})$

- $Cl_{1,0} = \mathbb{C}$ & $Cl_{0,1} = \mathbb{R} \oplus \mathbb{R}$
- $f : Cl_n \xrightarrow{\cong} Cl_{n+1}^{0}$ ($0 \leftarrow$ even) s.t.

$$f|_{Cl_n^0} = 1 \quad , \quad f|_{Cl_n^1} = e_{n+1} \cdot$$

- $$Cl_n \xrightarrow{\cong} \Lambda^{\cdot}\mathbb{R}^n$$

$$v \cdot \varphi \longleftrightarrow v \wedge \varphi - v \lrcorner \varphi \quad \forall v \in \mathbb{R}^n$$

§

$\otimes_{\mathbb{R}}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{C}$	$2\mathbb{C}$	$\mathbb{C}(2)$
$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$

$\mathbb{R}(4) \equiv Mat_{4\times4}(\mathbb{R})$

- $$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathbb{C} \oplus \mathbb{C}$$

$$1 \otimes 1 + i \otimes i \quad (2 , 0)$$
$$1 \otimes 1 - i \otimes i \quad (0 , 2)$$

- $$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow End_{\mathbb{C}}(\mathbb{H}) = \mathbb{C}(2)$$

$$z \otimes q \longmapsto (x \longmapsto z x \bar{q})$$

- $$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \longrightarrow End_{\mathbb{R}}(\mathbb{H}) = \mathbb{R}(4)$$

$$q_1 \otimes q_2 \longmapsto (x \longmapsto q_1 x \bar{q}_2)$$

$$Cl_{1,0} = \mathbb{C} , \quad Cl_{2,0} = \mathbb{H}$$
$$Cl_{0,1} = 2\mathbb{R} , \quad Cl_{0,2} = \mathbb{R}(2) \equiv Mat_{2\times2}(\mathbb{R})$$

Thm. $Cl_{n,0} \otimes Cl_{0,2} = Cl_{0,n+2}$ & $Cl_{0,n} \otimes Cl_{2,0} = Cl_{n+2,0}$

Pf: $\underset{e_1',\dots,e_n',f_1',f_2'}{\mathbb{R}^{0,n+2}} \quad \underset{e_1,\dots,e_n}{\mathbb{R}^{n,0}} \quad \underset{f_1,f_2}{\mathbb{R}^{0,2}}$

$$e_j' \longmapsto e_j \otimes f_1 f_2$$
$$f_j' \longmapsto 1 \otimes f_j$$

(Use $f_1 f_2 f_1 f_2 = -1$ to sign on $\langle \ \rangle$.)

- $$Cl_{n+8,0} = Cl_{n,0} \otimes \underbrace{Cl_{0,2}}_{\mathbb{R}(2)} \otimes \underbrace{Cl_{0,2}}_{\mathbb{R}(2)} \otimes \underbrace{\underbrace{Cl_{2,0}}_{\mathbb{H}} \otimes \underbrace{Cl_{2,0}}_{\mathbb{H}}}_{\mathbb{R}(4)}$$
  $$= Cl_{n,0} \otimes \mathbb{R}(16)$$ periodicity !

- $Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}l_{r+s}$ ($\because \nexists$ signature for $q_{\mathbb{C}}$)

- $\mathbb{C}l_{n+2} = \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}(2)$ cpx. periodicity.

§ Theorem (1) $\mathbb{R}(n) \curvearrowright \mathbb{R}^n$ (or $/\mathbb{C}$, $/\mathbb{H}$) is the only irred. real alg repr.

(2) $\mathbb{R}(n) \oplus \mathbb{R}(n)$ has exactly 2 such, namely from proj. to 2 factors.

n	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$2\mathbb{R}(8)$	$\mathbb{R}(16)$
#irred rep.	1	1	2	1	1	1	2	1
$\$/\$_{\pm}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}^2$	$\mathbb{C}^4$	$\mathbb{R}^8$	$\mathbb{R}^8$	$\mathbb{R}^{16}$

Rest follows from periodicity.

- $/\mathbb{C}$

$Cl^0_{2m+1} = \mathbb{C}l_{2m} = \mathbb{C}(2^m)$ $\qquad \$^{\mathbb{C}}_{2m+1} = \mathbb{C}^{2^m}$

$Cl^0_{2m+2} = \mathbb{C}l_{2m+1} = 2\,\mathbb{C}(2^m)$ $\qquad \$^{\mathbb{C}}_{\pm,2m+2} = \mathbb{C}^{2^m}$

§ $\Lambda^\bullet \mathbb{R}^n \xrightarrow[\simeq]{\text{vector space}} Cl_n$

$$\begin{array}{ccccc} & \Lambda^\bullet \mathbb{R}^n & & Cl_n & \\ & \cup\!\!\mid & & \cup\!\!\mid & \\ so(n) \simeq & \Lambda^2 \mathbb{R}^n & \longleftrightarrow & spin(n) & \\ & 2\, e_i \wedge e_j & \longleftrightarrow & e_i \cdot e_j & \\ (& v \wedge w & \longleftrightarrow & \frac{1}{4}[v, w] & \text{in general}) \end{array}$$

§ Application

$$Cl_n \curvearrowright \mathbb{R}^{N+1}$$

$\Longrightarrow \exists\, n$ ptwise linear indep. vector fields on S^N
(and $\mathbb{RP}^N$)

(via $v_i(x) = e_i \cdot x$)

$$n_{max} = 8a + 2^b - 1 \quad \text{if} \quad N+1 = 2^{4a+b}(2t+1)$$

($\exists$ so many : by finding repr. (eg periodicity ~ a)
$\nexists$ no more : theorem of Adams.

- Similar for $S^N/\mathbb{Z}_p$ w/ $\mathbb{Z}_p \leqslant U(1)$
  S^N/Γ w/ $\Gamma \leqslant Sp(1) = SU(2)$

and $\mathbb{CP}^n$ $\mathbb{HP}^n$

§ Applications to Lie theory

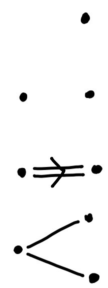
$$\mathbb{H}^n = \mathbb{C}^{2n} = \mathbb{R}^{4n}$$

$$Sp(n) \subset SU(2n)$$

$$U(n) \subset SO(2n)$$

dim $\quad n(n+1) \qquad n \qquad \frac{1}{2}m(m-1)$ $\quad m=2n$

Thm.
$$\begin{aligned} Spin(3) &= SU(2) = Sp(1) \\ Spin(4) &= Sp(1)^2 \\ Spin(5) &= Sp(2) \\ Spin(6) &= SU(4) \end{aligned}$$



(view from Dynkin diagram)

Thm. $Spin(7) \curvearrowright \mathbb{R}^8 = \mathbb{O}$ transitive on S^7

In particular, $S^7 = Spin(7)/G_2$

Thm. $Spin(8) \curvearrowright \$_+, \$_-, \mathbb{R}^8 \rightsquigarrow$ triality

Thm. $Spin^\circ(2,1) = SL(2,\mathbb{R})$

$Spin^\circ(3,1) = SL(2,\mathbb{C}) \qquad Spin^\circ(2,2) = SL(2,\mathbb{R})^2$

$\underbrace{Spin^\circ(5,1)}_{\Uparrow} = SL(2,\mathbb{H}) \qquad Spin^\circ(3,3) = \widetilde{SL}(4,\mathbb{R})$

$Spin^\circ(\mathbb{A} \oplus \mathbb{R}^{1,1})$

w/ $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

§ K-theory & Atiyah-Bott-Shapiro Constr.

$\mathrm{Vect}(X) := \{ VB/X \}/\text{isom.}$, X compact

$\rightsquigarrow K(X) \ni V - W$ Virtual Vector Bundle

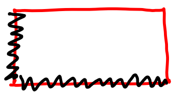
$(K(X), \oplus, \otimes)$: ring (like $\mathbb{N} \rightsquigarrow \mathbb{Z}$)

- $0 \to \tilde{K}(X) \to K(X) \to K(pt) \to 0$

  reduced K-gp. , i.e. $\mathrm{rk}\, V = \mathrm{rk}\, W$.

- $Y \overset{\text{closed}}{\subset} X \rightsquigarrow$ relative K-group $K(X,Y) \triangleq \tilde{K}(X/Y)$

- Higher K-group $K^{-i}(X) \triangleq \tilde{K}(\underbrace{S^i \wedge X^+}_{\frac{S^i \times X}{S^i \times p \,\cup\, p' \times X}})$ smash product



- $\cdots \to K^{-i}(X,Y) \to K^{-i}(X) \to K^{-i}(Y) \xrightarrow{\delta} K^{-i+1}(X,Y) \to \cdots$

- $K^{-i}(X_1) \otimes K^{-j}(X_2) \longrightarrow K^{-i-j}(X_1 \wedge X_2)$

$\rightsquigarrow K^{-\bullet}(pt)$ graded ring ; $K^{-\bullet}(X)$ graded mod/it.

Bott periodicity

(1) $K^{-\bullet}(pt) \simeq \mathbb{Z}[\xi]$ w/ $\xi = \mathcal{O}(1) - \underline{\mathbb{C}} \in \tilde{K}(S^2) = K^{-2}(pt)$

(2) $K^{-i}(X) \xrightarrow[\cong]{\xi \otimes} K^{-i-2}(X)$

Bott periodicity /$\mathbb{R}$

(1) $KO^{-\bullet}(pt) \simeq \mathbb{Z}[\overset{\text{deg } -2}{\eta}, \overset{-4}{y}, \overset{-8}{x}]/\langle 2\eta, \eta^3, \eta y, y^2 - 4x \rangle$

(2) $KO^{-i}(X) \xrightarrow[\cong]{x \otimes} K^{-i-8}(X)$

Reformulation of Bott periodicity:

$$K_{cpt}(X) \simeq K_{cpt}(X \times \mathbb{C}) \qquad \text{K-th. w/ cpt supp.}$$

$$KO_{cpt}(X) \simeq KO_{cpt}(X \times \mathbb{R}^8)$$

where $K_{cpt}(X) \ni [E_0, E_1, \mu]$

w/ $\mu : E_0|_{X \setminus cpt} \xrightarrow{\cong} E_1|_{X \setminus cpt}$

- $Cl_n \curvearrowright W = W^0 \oplus W^1$

$\rightsquigarrow [E_0, E_1, \mu] \in K(D^n, S^{n-1}) \simeq \tilde{K}(S^n) = K^{-n}(pt)$

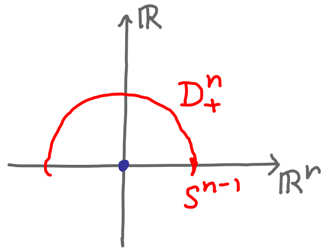
$E_0 = \underline{\underline{W^0}}, \qquad E_1 = \underline{\underline{W^1}}$

$$\mu : E_0|_{S^{n-1}} \longrightarrow E_1|_{S^{n-1}}$$

$$\begin{array}{ccc} W & \longmapsto & x \cdot W \\ \downarrow & & \downarrow \\ S^{n-1} \ni x & & x \end{array}$$

- If $Cl_n \leq Cl_{n+1} \curvearrowright W$

$\Longrightarrow \mu$ can be extended to D^n



$\Longrightarrow [E_0, E_1, \mu] = 0 \in K(D^n, S^{n-1})$

$$\rightsquigarrow \frac{\{\text{virtual } Cl_* \text{-mod}\}}{\{\text{virtual } Cl_{*+1}\text{mod}\}} \longrightarrow K^{-*}(pt)$$

ring homo.

$\Downarrow$

Atiyah-Bott-Shapiro: isomorphism

- Pf. did use Bott's result.
- LHS easy to identify ($\Rightarrow$ Bott periodicity)
- Similar for KO